Non-Gaussian two-mode squeezing and continuous variable entanglement of linearly and circularly polarized light beams interacting with cold atoms

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We investigate how entangled coherent states and superpositions of low intensity coherent states of non-Gaussian nature can be generated via non-resonant interaction between either two linearly or circularly polarized field modes and an ensemble of X-like four-level atoms placed in an optical cavity. We compare our results to recent experimental observations and argue that the non-Gaussian structure of the field states may be present in those systems.

I. INTRODUCTION

Quantum state entanglement is a recognized resource for achieving efficient quantum communication protocols. With the emergence of proposals for continuous variable (CV) quantum communication protocols [1, 2], such as quantum teleportation [3, 4] and dense coding [5, 6, 7], there has been an increasing interest in the generation and manipulation of entanglement in the CV regime through a diversity of experimental setups. Recently, bipartite CV entanglement was achieved in a most remarkable experiment, through the interaction of a coherent linearly polarized light beam with a cloud of cold atoms in a high finesse optical cavity [8]. The end product of the interaction is the generation of two entangled squeezed modes with orthogonal polarizations. The entanglement between those beams is demonstrated by checking the inseparability criterion for CV [9, 10]. The inseparability criterion is strictly a *necessary and sufficient condition* for entanglement only for Gaussian states, but is a *sufficient condition* for entanglement for any other CV state. However it is not completely evident whether the bipartite state generated in this experiment is Gaussian or not. Precisely speaking the interaction between the two orthogonally polarized fields intermediated by the atomic cloud is highly expected to be nonlinear in the field modes annihilation and creation operators. As is well known only Hamiltonians which are at most bilinear in canonically conjugated variables can lead to Gaussian evolution [11]. A deeper analysis of this system was given in a series of papers [12, 13, 14], and some results were numerically confirmed [15]. However it would be certainly important to stress all the available possibilities for generating entanglement and to infer on the Gaussian or non-Gaussian character in this experimental setup.

In this paper we investigate a model [12] for the interaction of two orthogonally polarized quantum fields with an ensemble of X-like four level atoms, deriving an effective Hamiltonian accounting for the field modes interaction. Up to first order the interaction results to be bilinear in the field operators, which however is multiplied by the difference of population of an effective ensemble of two-level atoms, possibly leading thus to a non-Gaussian evolution. Conditioned on the atomic population measurement, the two orthogonally polarized fields are left on an non-Gaussian entangled state, which shows similar properties of the Gaussian two-mode squeezed vacuum state. Since the experiment allows for two regimes depending on the detuning between the incident light and the atomic system, called self-rotation [16] or polarization switching [8], one deals either with a circularly polarized beam or a squeezed linearly polarized beam. By appropriately setting one of the input linearly polarized modes in the vacuum state, we obtain in the output a coherent superposition of coherent states in the orthogonally polarized linear modes. When viewed from the circular polarization frame, this superposition results in an entangled coherent state between the two modes in polarization + and −. The presence of this superposition, in one polarization reference frame and an entangled state in the other can explain all the non-classical features observed in the before mentioned experiment, with a non-Gaussian state however. Remarkably recently much effort has been dedicated to the generation of such non-Gaussian states in propagating light fields by photon-subtraction [17, 18]. In this paper we show that in principle such states may be generated in the experiments reported in [8, 12, 13, 14] as well.

This paper is organized as follows. In Sec. II we review the description of the interaction between the two orthogonally polarized fields with an ensemble of N cold atoms following [13] and obtain an approximate solution to the Heisenberg-Langevin equations that govern the atomic ensemble evolution in the dispersive regime considered. We then derive an effective Hamiltonian accounting for the interaction of an ensemble of two-level atoms and fields in both circular and linear polarizations. We investigate, in Sec. III, the dynamical generation of entangled coherent states, and conditional generation of superposition states. In Sec. IV we analyze the squeezing of quadratures variances for both linear and circular polarizations and infer on the inseparability of the two orthogonal modes. Finally, Section V contains a summary and conclusions.
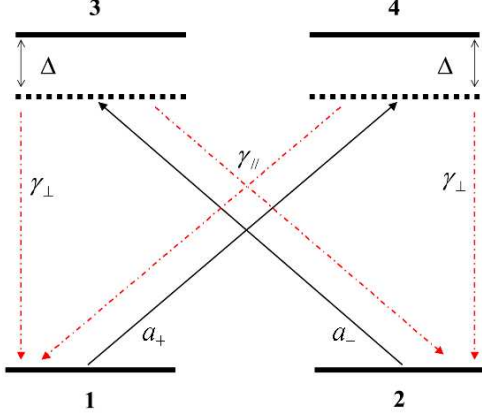
FIG. 1: (Color online) Scheme of the four level X-like atomic model for the $6S_{1/2}, F = 4$ to $6P_{3/2}, F = 5$ transition of Cesium.

II. THE MODEL

Light fields interacting with cold atoms can show a diversity of interesting phenomena such as squeezing, two-mode squeezing and CV entanglement [8, 12, 13, 14]. It can also be employed for implementation of quantum logic operations [19]. Recent experiments have demonstrated the direct relation between one mode squeezing and two-mode entanglement under a linear (Bogoliubov) transformation of the fields polarization reference frame [13, 14]. In those experiments, an x-linearly polarized probe field is let to interact with a cloud of cold cesium atoms in a high finesse optical cavity. The cavity output x-polarized signal and the y-polarized vacuum are squeezed, which results in the entanglement of two orthogonal circular polarization fields. The probe light field is red-detuned by about 50 MHz of the $6S_{1/2}, F = 4$ to $6P_{3/2}, F = 5$ transition. As previously argued [12] this complicated transition can be modeled by an X-like four-level atomic structure.

Following Ref.[12] we consider the atomic system as being a set of N X-like four-level cold atoms in an optical cavity driven by a linearly polarized field, as shown schematically in figure 1. We employ collective operators to describe the N atoms ensemble (e.g. $\sigma_{14} = \sum_{i=1}^{N} e^{i\omega t}|1\rangle_i\langle 4|_i$), and we denote the operators related to circularly polarized fields by indexes $a_{+(-)}$, which are defined from the standard linear polarization components

$$a_+ = \frac{a_x - ia_y}{\sqrt{2}} \qquad \text{and} \qquad a_- = \frac{a_x + ia_y}{\sqrt{2}} \,. \qquad (1)$$

We shall consider both orthogonally polarized linear modes a_x and a_y initially in coherent states and latter we assume the mode a_y in its vacuum state. The atomic transition frequencies are chosen in resonance ($\omega_{at} = \omega_{13} = \omega_{24}$) for simplification matters. In that case, if the field

frequency is ω, the detuning from the atomic ensemble transitions resonance is equal to $\Delta = \omega_{at} - \omega$. The coupling constant between the atoms and field is $g = \epsilon_0 d/\hbar$, where d is the atomic dipole and $\epsilon_0 = \sqrt{\hbar\omega/2\varepsilon V}$, where V is the volume of the cavity. The dipole decay rate γ, as depicted in fig. 1, is decomposed in two orthogonal rates as $\gamma = \gamma_\| + \gamma_\perp$. Thus, the atoms-field interaction Hamiltonian is described by [20]

$$H = H_0 + H_I \qquad (2)$$

where

$$
\begin{aligned}
H_0 &= E_1\sigma_{11} + E_2\sigma_{22} + E_3\sigma_{33} + E_4\sigma_{44} \\
&\quad + \hbar\omega a_+^\dagger a_+ + \hbar\omega a_-^\dagger a_-,
\end{aligned} \qquad (3)
$$

and

$$
\begin{aligned}
H_I &= \hbar g \Big[e^{-i\omega t}a_+\sigma_{41} + e^{i\omega t}a_+^\dagger\sigma_{14} \\
&\quad + e^{-i\omega t}a_-\sigma_{32} + e^{i\omega t}a_-^\dagger\sigma_{23} \Big].
\end{aligned} \qquad (4)
$$

The atomic evolution is appropriately governed by a set of quantum Heisenberg-Langevin equations [12], here given in a rotating frame with the probe frequency ω as

$$\dot{\sigma}_{14} = -(\gamma + i\Delta)\sigma_{14} - iga_+(\sigma_{11} - \sigma_{44}) + F_{14}, \qquad (5)$$

$$\dot{\sigma}_{23} = -(\gamma + i\Delta)\sigma_{23} - iga_-(\sigma_{22} - \sigma_{33}) + F_{23}, \qquad (6)$$

$$\dot{\sigma}_{11} = 2\gamma_\perp\sigma_{33} + 2\gamma_\|\sigma_{44} - ig(a_+^\dagger\sigma_{14} - h.c.) + F_{11}, (7)$$

$$\dot{\sigma}_{22} = 2\gamma_\|\sigma_{33} + 2\gamma_\perp\sigma_{44} - ig(a_-^\dagger\sigma_{23} - h.c.) + F_{22}, (8)$$

$$\dot{\sigma}_{33} = -2\gamma\sigma_{33} + ig(a_-^\dagger\sigma_{23} - h.c.) + F_{33}, \qquad (9)$$

$$\dot{\sigma}_{44} = -2\gamma\sigma_{44} + ig(a_+^\dagger\sigma_{14} - h.c.) + F_{44}, \qquad (10)$$

(with $\hbar = 1$) where F_{ij} are the Langevin operators.

We want to derive an effective Hamiltonian accounting for the interaction between two- orthogonally polarized field modes. For that we make the following assumptions:

(i) We shall neglect the fluctuation on deriving a stationary solutions to the above set of equations. That means that we shall neglect the Langevin operators by taking $F_{ij} = 0$.

(ii) To simplify the four level system of atoms to an effective system of two level atoms, we consider that Δ is very large ($\Delta \gg \gamma \gg g$) in such a way that the higher levels 3 and 4 are not significantly populated ($\sigma_{33} = \sigma_{44} = 0$). Thus by taking Eqs. (5,6) in the stationary regime ($\dot{\sigma}_{14} = \dot{\sigma}_{23} = 0$) results in (0) lowest order approximation

$$\sigma_{14}^{(0)} = \frac{-iga_+(t)\sigma_{11}(t)}{(\gamma + i\Delta)}, \qquad (11)$$

and

$$\sigma_{23}^{(0)} = \frac{-iga_-(t)\sigma_{22}(t)}{(\gamma + i\Delta)}, \qquad (12)$$

These solutions can be replaced back into Eqs. (7-10) for the levels population. Rewriting the interaction term in a symmetrized form we obtain

$$\dot{\sigma}_{44} = \frac{2\gamma g^2}{(\gamma^2 + \Delta^2)}(a_+^\dagger a_+ + 1/2)\sigma_{11}$$
$$-2\gamma \left[1 + \frac{g^2}{(\gamma^2 + \Delta^2)}(a_+^\dagger a_+ + 1/2)\right]\sigma_{44}, \quad (13)$$

$$\dot{\sigma}_{33} = \frac{2\gamma g^2}{(\gamma^2 + \Delta^2)}(a_-^\dagger a_- + 1/2)\sigma_{22}$$
$$-2\gamma \left[1 + \frac{g^2}{(\gamma^2 + \Delta^2)}(a_-^\dagger a_- + 1/2)\right]\sigma_{33}, \quad (14)$$

$$\dot{\sigma}_{22} = 2\gamma_\perp \sigma_{44} - \frac{2\gamma g^2}{(\gamma^2 + \Delta^2)}(a_-^\dagger a_- + 1/2)\sigma_{22}$$
$$+2\gamma \left[\frac{\gamma_\parallel}{\gamma_\perp} + \frac{g^2}{(\gamma^2 + \Delta^2)}(a_-^\dagger a_- + 1/2)\right]\sigma_{33}, (15)$$

$$\dot{\sigma}_{11} = 2\gamma_\perp \sigma_{33} - \frac{2\gamma g^2}{(\gamma^2 + \Delta^2)}(a_+^\dagger a_+ + 1/2)\sigma_{11}$$
$$+2\gamma \left[\frac{\gamma_\parallel}{\gamma_\perp} + \frac{g^2}{(\gamma^2 + \Delta^2)}(a_+^\dagger a_+ + 1/2)\right]\sigma_{44}, (16)$$

where we have already neglected the Langevin terms. In the stationary regime where the states 3 and 4 will be considered practically not populated we find

$$\sigma_{33} = \frac{g^2}{(\gamma^2 + \Delta^2)}\left[1 + \frac{g^2}{(\gamma^2 + \Delta^2)}(a_-^\dagger a_- + 1/2)\right]^{-1}$$
$$\times(a_-^\dagger a_- + 1/2)\sigma_{22}, \quad (17)$$

$$\sigma_{44} = \frac{g^2}{(\gamma^2 + \Delta^2)}\left[1 + \frac{g^2}{(\gamma^2 + \Delta^2)}(a_+^\dagger a_+ + 1/2)\right]^{-1}$$
$$\times(a_+^\dagger a_+ + 1/2)\sigma_{11}. \quad (18)$$

Since we have assumed $\Delta \gg \gamma \gg g$, we must have $\frac{g^2}{(\gamma^2+\Delta^2)} \ll 1$, and from now on we shall only keep in (17) and (18) first order terms in this quantity. That is, we assume

$$\sigma_{33} \approx \frac{g^2}{(\gamma^2 + \Delta^2)})(a_-^\dagger a_- + 1/2)\sigma_{22}, \quad (19)$$

$$\sigma_{44} \approx \frac{g^2}{(\gamma^2 + \Delta^2)}(a_+^\dagger a_+ + 1/2)\sigma_{11}. \quad (20)$$

These stationary solutions for the populations of levels 3 and 4 can be then included in the correspondent first order terms for the coherences between sates 1,4 and 2,3 are as follows

$$\sigma_{14}^{(1)} = \frac{-iga_+(t)(\sigma_{11} - \sigma_{44})}{(\gamma + i\Delta)}, \quad (21)$$

and

$$\sigma_{23}^{(1)} = \frac{-iga_-(t)(\sigma_{22} - \sigma_{33})}{(\gamma + i\Delta)}. \quad (22)$$

We now are in position to consider the Heisenberg equations for the field operators

$$\dot{a}_+ = -i\omega a_+ - ig\boldsymbol{\sigma}_{14}, \quad (23)$$
$$\dot{a}_- = -i\omega a_- - ig\boldsymbol{\sigma}_{23}, \quad (24)$$

where $\boldsymbol{\sigma}_{14(23)} = e^{i\omega t}\sigma_{14(23)}$. Substituting the stationary solutions (21) and (22), we obtain

$$\dot{a}_+ = -\left\{i\omega + \frac{g^2}{2(\gamma + i\Delta)}\left[1 - \frac{g^2 a_+ a_+^\dagger}{(\gamma^2 + \Delta^2)}\right]\right\}a_+$$
$$-\frac{g^2}{2(\gamma + i\Delta)}\left[1 - \frac{g^2 a_+ a_+^\dagger}{(\gamma^2 + \Delta^2)}\right]a_+\boldsymbol{\sigma}_z, \quad (25)$$

$$\dot{a}_- = -\left\{i\omega + \frac{g^2}{2(\gamma + i\Delta)}\left[1 - \frac{g^2 a_- a_-^\dagger}{(\gamma^2 + \Delta^2)}\right]\right\}a_-$$
$$+\frac{g^2}{2(\gamma + i\Delta)}\left[1 - \frac{g^2 a_- a_-^\dagger}{(\gamma^2 + \Delta^2)}\right]a_-\boldsymbol{\sigma}_z, \quad (26)$$

where we have defined $\boldsymbol{\sigma}_z \equiv \boldsymbol{\sigma}_{11} - \boldsymbol{\sigma}_{22}$, and we assumed $\boldsymbol{\sigma}_{11} + \boldsymbol{\sigma}_{22} \approx \mathbf{1}$. Remark the presence of the non-linear terms proportional to $a_+ a_+^\dagger a_+$ and to $a_- a_-^\dagger a_-$. They will lead to non-Gaussian evolutions over the field modes, i.e., if the initial field mode states are Gaussian they will be driven to non-Gaussian states. The same is true for higher order terms, which imply that invariably the field operators evolution will be non-Gaussian as inferred in the introduction. These nonlinear terms are at least proportional to g^4, which from our assumptions is very small compared to the linear terms in g^2, which alone may or not lead to Gaussian evolutions depending on the prepared atomic system state. Keeping only the terms up to linear in g^2, Eqs. (25) and (26) simplify to

$$\dot{a}_+ \approx -\left[i\omega + \frac{g^2}{2(\gamma + i\Delta)}\right]a_+ - \frac{g^2}{2(\gamma + i\Delta)}a_+\boldsymbol{\sigma}_z \quad (27)$$

$$\dot{a}_- \approx -\left[i\omega + \frac{g^2}{2(\gamma + i\Delta)}\right]a_- + \frac{g^2}{2(\gamma + i\Delta)}a_-\boldsymbol{\sigma}_z, (28)$$

which, in terms of the linear polarization operators are given by

$$\dot{a}_x \approx -\left[i\omega + \frac{g^2}{2(\gamma + i\Delta)}\right]a_x + \frac{ig^2}{2(\gamma + i\Delta)}a_y\boldsymbol{\sigma}_z, (29)$$

$$\dot{a}_y \approx -\left[i\omega + \frac{g^2}{2(\gamma + i\Delta)}\right]a_y - \frac{ig^2}{2(\gamma + i\Delta)}a_x\boldsymbol{\sigma}_z. (30)$$

Assuming $\mathbf{a}_i \equiv e^{\left(i\omega + \frac{g^2}{2(\gamma + i\Delta)}\right)t}a_i$, $i = x, y$ we can rewrite these equations as

$$\dot{\mathbf{a}}_x = \frac{ig^2}{2(\gamma + i\Delta)}\mathbf{a}_y\boldsymbol{\sigma}_z \quad \text{and} \quad \dot{\mathbf{a}}_y = -\frac{ig^2}{2(\gamma + i\Delta)}\mathbf{a}_x\boldsymbol{\sigma}_z.(31)$$

The appropriate effective interaction Hamiltonians corresponding to the evolution of the circularly polarized modes corresponding to Eqs. (27,28) is

$$H_{eff}^{(c)} = -i\lambda(\mathbf{a}_+^\dagger \mathbf{a}_+ - \mathbf{a}_-^\dagger \mathbf{a}_-)\boldsymbol{\sigma}_z, \quad (32)$$

with

$$\lambda = \frac{g^2}{2(\gamma + i\Delta)} \equiv \lambda_1 + i\,\lambda_2, \qquad (33)$$

where

$$\lambda_1 \equiv \frac{g^2\gamma}{2(\gamma^2 + \Delta^2)}, \qquad (34)$$

$$\lambda_2 \equiv -\frac{g^2\Delta}{2(\gamma^2 + \Delta^2)}. \qquad (35)$$

Similarly for the linear polarization with respect to Eqs. (31), we obtain

$$H_{eff}^{(l)} = \lambda(\mathbf{a}_y^\dagger \mathbf{a}_x - \mathbf{a}_x^\dagger \mathbf{a}_y)\boldsymbol{\sigma}_z, \qquad (36)$$

which is a bilinear coupling between the x and y polarization modes. Had we considered the nonlinear terms as discussed above we would end up with nonlinear interaction term as well, such as $a_x^\dagger a_x a_y^\dagger a_y$, which would drive the system state to a non-Gaussian entangled state such as discussed in [21].

Remark that, being λ a complex number, both effective Hamiltonians are non-Hermitian. The non-Hermiticity comes from the spontaneous emission from states 3 and 4 and from the fact that we have eliminated those states in our treatment, leading to non-unitary process. Our choice for the approximations ($\Delta \gg \gamma \gg g$) are in accordance with experimental values, which settle $\gamma \approx 2.6-16$ MHz, $\Delta \approx 130-327$ MHz, and g around 2 Hz. To keep such relations between the parameters (Δ, γ, g), the appropriate relation for the real and imaginary part of the coupling of the effective Hamiltonians is $|\lambda_2| \gg |\lambda_1|$. Thus the effective Hamiltonians assume the following form

$$H_{eff}^{(c)} = \lambda_2(\mathbf{a}_+^\dagger \mathbf{a}_+ - \mathbf{a}_-^\dagger \mathbf{a}_-)\boldsymbol{\sigma}_z \qquad (37)$$

and

$$H_{eff}^{(l)} = i\lambda_2(\mathbf{a}_y^\dagger \mathbf{a}_x - \mathbf{a}_x^\dagger \mathbf{a}_y)\boldsymbol{\sigma}_z, \qquad (38)$$

III. ENTANGLED COHERENT STATES AND SUPERPOSITIONS OF COHERENT STATES

We shall take firstly the Hamiltonians (32) and (36) in full form and then we consider the limit $|\lambda_1| \ll |\lambda_2|$ to properly set the time scale. As we shall see shortly, these Hamiltonians will generate non-Gaussian states only in favorable situations for the population imbalance of the atomic ensemble. To take into account the possible transitions between the two fundamental collective atomic states, we consider one of the two possibilities:

(a) The ensemble of atoms is found in a coherent macroscopic superposition of states 1 and 2, $|\chi_a^\pm\rangle = (1/\sqrt{2})(|1\rangle_{at} \pm |2\rangle_{at})$, where $|1(2)\rangle_{at} \equiv |11..1(22..2)\rangle$.

(b) Each atom of the ensemble is found in a coherent superposition of states 1 and 2: $|\chi'_a\rangle = \Pi_{i=1}^N \frac{1}{\sqrt{2}}(|1\rangle_i + |2\rangle_i)$.

Ensembles of atoms prepared in coherent superposition of states interacting with light are important for several applications in quantum optics and quantum communication [22, 23, 24, 25, 26, 27, 28] and constitute an essential resource for generation of entanglement for the kind of light field states we analyze. Although the first situation (a) is not experimentally easy to achieve, it will illustrate more easily the final non-Gaussian nature of the field states. The second situation (b) is possible to be realized by applying non resonant classical light π-pulses to the atomic cloud, initially prepared in one of its ground states, before the interaction with the quantum field polarization we are considering. Alternatively one could employ more recent techniques to generate the coherent superposition, such as the one proposed in Ref. [29]. The field state resulting in this case is a generalization of the previous one, also clearly non-Gaussian.

(i) Circular Polarization. We consider both circularly polarized modes initially prepared in coherent states ($|\alpha\rangle_+$ and $|\beta\rangle_-$), where the subscript $+$ or $-$ designate the two orthogonal circular polarizations. Firstly we assume the atomic ensemble state (a) After some calculation the time evolved atoms-field state is

$$\begin{aligned}
|\psi(t)\rangle_c &= \frac{1}{\sqrt{2}}[|\alpha e^{\lambda t}\rangle_+ |\beta e^{-\lambda t}\rangle_- |1\rangle_{at} \\
&\pm |\alpha e^{-\lambda t}\rangle_+ |\beta e^{\lambda t}\rangle_- |2\rangle_{at}]
\end{aligned} \qquad (39)$$

which is obviously in an entangled state. Remark that the bilinear operation itself do not allow the two fields to be entangled if for example the atomic ensemble is found in one of the two states, $|1\rangle_{at}$ or $|2\rangle_{at}$, or a mixture of them. The atomic superposition of states is an essential resource to generate entanglement.

By conditioning the measurement of the atomic system in the same initial superposition (a), we find

$$\begin{aligned}
|\varphi(t)\rangle_\pm^c &= \frac{\langle \chi_a^\pm | \psi(t)\rangle}{\sqrt{Tr_{+,-}\{|\langle\chi_a^\pm|\psi(t)\rangle|^2\}}} \\
&= \frac{1}{\sqrt{N_\pm}}[|\alpha e^{\lambda t}\rangle_+ |\beta e^{-\lambda t}\rangle_- \pm |\alpha e^{-\lambda t}\rangle_+ |\beta e^{\lambda t}\rangle_-],
\end{aligned} \qquad (40)$$

where

$$\begin{aligned}
N_\pm &= 2\left\{1 \pm e^{-(|\alpha|^2+|\beta|^2)\cosh 2\lambda_1 t} e^{(|\alpha|^2+|\beta|^2)\cos 2\lambda_2 t}\right. \\
&\left. \times \cos[(|\alpha|^2 - |\beta|^2)\sin 2\lambda_2 t]\right\}
\end{aligned} \qquad (41)$$

Since λ has both real and imaginary parts we see that as the time evolves the two modes become more and more entangled, due both to an increase of the coherent states amplitudes and to a time dependent dephasing between the two components of the superposition. To illustrate

we set the time scale for $|\lambda_2|t = \pi/2$, obtaining

$$|\varphi(\pi/2|\lambda_2|)\rangle_\pm^c = \frac{1}{\sqrt{N_\pm}}[|e^{\frac{\pi\lambda_1}{2\lambda_2}}(i\alpha)\rangle_+|e^{-\frac{\pi\lambda_1}{2\lambda_2}}(-i\beta)\rangle_-$$
$$\pm|e^{-\frac{\pi\lambda_1}{2\lambda_2}}(-i\alpha)\rangle_+|e^{\frac{\pi\lambda_1}{2\lambda_2}}(i\beta)\rangle_-]. \quad (42)$$

Taking the $\lambda_1/|\lambda_2| \to 0$ limit in Eq. (42), $|\varphi(\pi/2|\lambda_2|)\rangle_\pm^c$ assumes the following form

$$\frac{1}{\sqrt{N_\pm}}[|(i\alpha)\rangle_+|(-i\beta)\rangle_- \pm |(-i\alpha)\rangle_+|(i\beta)\rangle_-], \quad (43)$$

which is an entangled, but *non-Gaussian* state.

To show that such non-Gaussian entangled state would be present in many experimental configurations let us consider the much simpler to achieve state (b). Following the same procedure, but for the second initial atomic ensemble state (b), we obtain after the conditioned measurement of the atomic system in the same initial ensemble of superposition states, the following state

$$|\varphi'(t)\rangle^c = \frac{\langle\chi'_a|\psi(t)\rangle}{\sqrt{Tr_{+,-}\{|\langle\chi'_a|\psi(t)\rangle|^2\}}} =$$
$$\frac{1}{\sqrt{N'(t)}}\left[\sum_{j=0}^N C_j^N|\alpha e^{\frac{N-2j}{N}\lambda t}\rangle_+|\beta e^{-\frac{N-2j}{N}\lambda t}\rangle_-\right], \quad (44)$$

where $C_j^N = \frac{N!}{j!(N-j)!}$ is the binomial coefficient, and $N'(t)$ the corresponding normalization factor. This is

a two mode non-Gaussian entangled superposition state with $N+1$ terms in the superposition. Despite being simpler to be experimentally realized this last state has a cumbersome structure than the previous case Eq. (40). For illustration we want to keep the simplest non-Gaussian state, and thus from now on we consider only the results for the coherent superposition state (a), but the results should follow in similar fashion for the experimentally more accessible situation (b).

(ii) Linear Polarization. Under the same initial conditions (both field modes in the circular polarization prepared in coherent states and the atoms in state (a), we consider the effect of the evolution operator

$$U^{(l)}(t) = \exp\left[-i\lambda(\mathbf{a}_y^\dagger\mathbf{a}_x - \mathbf{a}_x^\dagger\mathbf{a}_y)\boldsymbol{\sigma}_z t\right]. \quad (45)$$

for the linear polarization. This is a beam-splitter operation with a phase conditioning on the atomic state. It is well known [30, 31, 32] that the bilinear beam-splitter operation do not entangle classical states. As a consequence, if the states of mode (x) and (y) are prepared in coherent states they evolve as coherent non-entangled states, unless the atomic system is prepared in a superposition of the two ground states, 1 and 2.

The two polarization frames are related by the field operators from Eq. (1). By employing this relation it is immediate that the two initial states are related by

$$|\alpha\rangle_+|\beta\rangle_- \to |\alpha'\rangle_y|\beta'\rangle_x, \quad (46)$$

where $\alpha' \equiv i(\alpha-\beta)/\sqrt{2}$ and $\beta' \equiv (\alpha+\beta)/\sqrt{2}$. Under the evolution (45) the above state takes the form

$$|\psi(t)\rangle_\pm^l = \frac{1}{\sqrt{2}}U^{(l)}(t)D_y(\alpha')D_x(\beta')|0\rangle_y|0\rangle_x(|1\rangle_{at} \pm |2\rangle_{at})$$
$$= \frac{1}{\sqrt{2}}D_y\left(\cosh(\lambda t)\alpha' - i\sinh(\lambda t)\beta'\right)D_x\left(\cosh(\lambda t)\beta' + i\sinh(\lambda t)\alpha'\right)|0\rangle_y|0\rangle_x|1\rangle_{at}$$
$$\pm\frac{1}{\sqrt{2}}D_y\left(\cosh(\lambda t)\alpha' + i\sinh(\lambda t)\beta'\right)D_x\left(\cosh(\lambda t)\beta' - i\sinh(\lambda t)\alpha'\right)|0\rangle_y|0\rangle_x|2\rangle_{at}$$
$$= \frac{1}{\sqrt{2}}|\cosh(\lambda t)\alpha' - i\sinh(\lambda t)\beta'\rangle_y \otimes |\cosh(\lambda t)\beta' + i\sinh(\lambda t)\alpha'\rangle_x \ |1\rangle_{at}$$
$$\pm\frac{1}{\sqrt{2}}|\cosh(\lambda t)\alpha' + i\sinh(\lambda t)\beta'\rangle_y \otimes |\cosh(\lambda t)\beta' - i\sinh(\lambda t)\alpha'\rangle_x \ |2\rangle_{at}, \quad (47)$$

where $D_i(\alpha)$ is the displacement operator in α respective to the polarization $i = x, y$. Conditioning the atomic

detection into the same superposition state (a) prepared at the beginning, we have

$$|\varphi(t)\rangle_\pm^l = \frac{\langle\chi_a^\pm|\psi(t)\rangle}{\sqrt{Tr_{+,-}\{|\langle\chi_a^\pm|\psi(t)\rangle|^2\}}} = \frac{1}{\sqrt{N_\pm}}[|\cosh(\lambda t)\alpha' - i\sinh(\lambda t)\beta'\rangle_y \otimes |\cosh(\lambda t)\beta' + i\sinh(\lambda t)\alpha'\rangle_x$$
$$\pm|\cosh(\lambda t)\alpha' + i\sinh(\lambda t)\beta'\rangle_y \otimes |\cosh(\lambda t)\beta' - i\sinh(\lambda t)\alpha'\rangle_x].$$

Again for the choice $|\lambda_2|t = \pi/2$, we obtain

$$\begin{aligned}
|\varphi(\pi/2|\lambda_2|)\rangle^l_\pm &= \frac{1}{\sqrt{N_\pm}}\left|i\sinh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\alpha' + \cosh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\beta'\right\rangle_y \otimes \left|i\sinh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\beta' - \cosh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\alpha'\right\rangle_x \\
&\pm\frac{1}{\sqrt{N_\pm}}\left|i\sinh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\alpha' - \cosh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\beta'\right\rangle_y \otimes \left|i\sinh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\beta' + \cosh\left(\frac{\pi\lambda_1}{2\lambda_2}\right)\alpha'\right\rangle_x,
\end{aligned} \qquad (48)$$

which for $\lambda_1/|\lambda_2| \to 0$ turns out to be

$$\begin{aligned}
|\varphi(\pi/2|\lambda_2|)\rangle^l_\pm &= \frac{1}{\sqrt{N_\pm}}\left[\left|\frac{(\alpha+\beta)}{\sqrt{2}}\right\rangle_y\left|-i\frac{(\alpha-\beta)}{\sqrt{2}}\right\rangle_x \right.\\
&\left.\pm\left|-\frac{(\alpha+\beta)}{\sqrt{2}}\right\rangle_y\left|i\frac{(\alpha-\beta)}{\sqrt{2}}\right\rangle_x\right]. \quad (49)
\end{aligned}$$

To match further with the experimental conditions we set the x-polarized mode in a vacuum state, such that $\alpha - \beta = 0$, obtaining

$$|\varphi(\pi/2|\lambda_2|)\rangle^l_\pm = \frac{1}{\sqrt{N_\pm}}\left(\left|\sqrt{2}\alpha\right\rangle_y \pm \left|-\sqrt{2}\alpha\right\rangle_y\right)|0\rangle_x. \quad (50)$$

This state represents a superposition of coherent states for the y-polarized mode, namely the odd or even coherent state [33] for the + or - choice for the atomic state. This state is of a remarkable importance for many applications on quantum information and computation [34]. Now, in the circular polarization Eq. (43), the state follows

$$\begin{aligned}
|\varphi(\pi/2|\lambda_2|)\rangle^c_\pm &\approx \frac{1}{\sqrt{N_\pm}}[|(i\alpha)\rangle_+|(-i\alpha)\rangle_- \\
&\pm|(-i\alpha)\rangle_+|(i\alpha)\rangle_-], \quad (51)
\end{aligned}$$

and unlike the linear polarization case (superposition in one mode and vacuum in the orthogonal mode), the two orthogonal circular polarization modes are entangled. This is a particular realization of quasi-Bell states, the so-called entangled coherent states, as fully discussed in [36], and employed in [21].

To end this section we remark that through our calculations we have neglected dissipative effects over the two field modes. Obviously dissipative effects will be always present and will drive the system to mixed states, reducing the amount of entanglement. However, the inclusion of dissipative effects would not alter the non-Gaussian nature of the field modes. Thus, since we want to keep the evolution as close as possible from a Gaussian one, and to simplify our discussion, we will not consider the effects of dissipation for the field modes.

IV. QUADRATURE VARIANCES AND ENTANGLEMENT CRITERIA

Entanglement in the circular polarization can be inferred from non-classicality signatures on the linear polarization (see e.g. [32, 35]). This is commonly realized by analyzing squeezing of the quadratures variances, as in Ref. [13]. Employing the usual definition of the quadrature operators

$$X_l = \frac{1}{2}(\mathbf{a}_l + \mathbf{a}_l^\dagger) \qquad Y_l = \frac{1}{2i}(\mathbf{a}_l^\dagger - \mathbf{a}_l), \quad (52)$$

where $l = +, -, x,$ or y for each of the circular or linear polarization mode, the variance for the X_l quadrature is given by $(\Delta X_l)^2 = \langle X_l^2\rangle - \langle X_l\rangle^2$, and similarly for Y_l. These two quadratures in a polarization frame can be combined to indicate if there is entanglement in another polarization frame through the CV inseparability criterion [9, 10]. In terms of variance operators it is given by

$$I_{a,b} = \frac{1}{2}[\Delta^2(X_a + X_b) + \Delta^2(Y_a - Y_b)] < 1. \quad (53)$$

For Gaussian states, $I_{a,b} < 1$ is a necessary and sufficient condition for entanglement. In our case where the states are non-Gaussian, while $I_{a,b} < 1$ undoubtedly indicates entanglement, when $I_{a,b} \geq 1$ nothing can be said about the presence of entanglement. In this section we analyze the behavior of the quadratures variances as indicators of entanglement and the inseparability criteria for the non-Gaussian states obtained. These results should be compared to the ones of Fig. 2 from Ref. [13], attributed to Gaussian states.

(i) Quadrature variances for the circularly polarized modes. The quadrature variances calculated from the state of Eq. (40), and with parameters fixed to reproduce the experimental situation [12] in the case the x-polarized mode is in a vacuum state ($\alpha = \beta$) are depicted in Fig 2. In Fig.2(a) we see an oscillatory (periodic) behavior for both quadratures variances of the + circularly polarized mode. Notice that the variance for the quadrature Y_+ is periodically compressed, oscillating bellow the reference line at the value $1/4$, while the variance for X_+ always oscillates above this line. In Fig.2(b), the only modification relative to the previous case is the value of the ratio between the λ's ($\lambda_1/|\lambda_2| = 0.1$) In that case the variance of the quadrature X_+ is compressed as well with time. We see a similar behavior, but for longer times the amplitude of the oscillations increases due to the non-Hermitian nature of the Hamiltonian. The variances X_- and Y_- show exactly the same behavior of the ones for the + polarization.
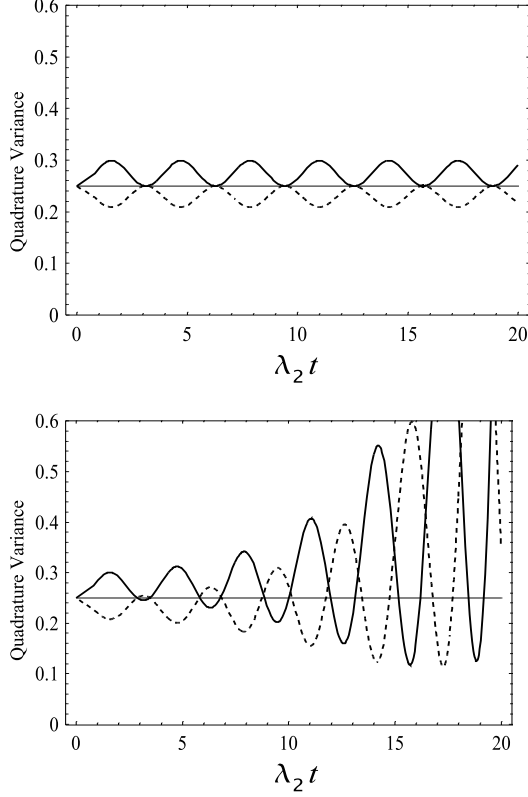
FIG. 2: Time evolution of the quadrature variances for circularly polarized modes, with $\beta = \alpha$, $Re[\alpha] = 0$ and $Im[\alpha] = 0.3$. (a) $(\Delta X_+)^2$ (continuous line) and $(\Delta Y_+)^2$ (dashed line) for $\lambda_1/|\lambda_2| = 0$. (b) $(\Delta X_+)^2$ (continuous line) and $(\Delta Y_+)^2$ (dashed line) for $\lambda_1/|\lambda_2| = 0.1$.

For the linearly polarized modes the criteria writes

$$I_{x,y}(\alpha,\beta) = \frac{1}{2}[\Delta^2(X_x + X_y)(\alpha,\beta) \\ + \Delta^2(Y_x - Y_y)(\alpha,\beta)] < 1, \qquad (54)$$

and for the situation $\alpha = \beta$ the inequality correctly indicates no entanglement, since the x mode is left in a vacuum state. The plots for this case are shown in Fig.3. Fig.3(a) is for $\lambda_1/|\lambda_2| = 0$, while fig. 3(b) is for $\lambda_1/|\lambda_2| = 0.1$. In all situations the inequality is violated, which in this case correctly indicates no entanglement.

(ii) Quadrature variances for the linearly polarized modes. For the linearly polarized modes we shall consider the variances from the state of Eq. (40) with $\alpha = \beta$, and $\lambda_1/|\lambda_2| \to 0$. As can be noted in the plots of Fig.4(a) for this case and $\lambda_1/|\lambda_2| = 0$, only the variance of the X_y quadrature is squeezed (always below the reference line), while the x-mode is left in a vacuum state. Again in Fig. 4(b) we take $\lambda_1/|\lambda_2| = 0.1$ and remark that the variance of the quadrature Y_y is squeezed as well with time as an effect of the non-Hermiticity.

Since the x polarized mode is always in a vacuum state the variances of its quadratures will not change with time.
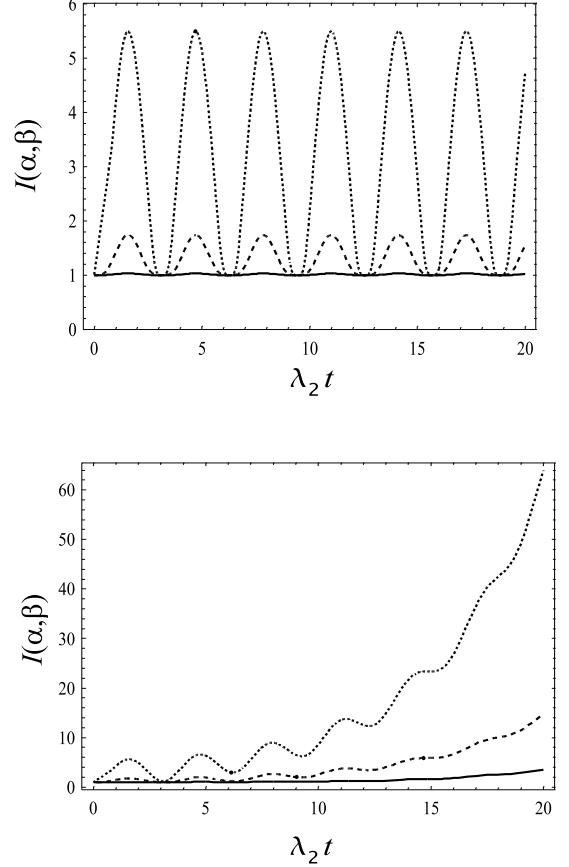

FIG. 3: Plot of the time evolution of the inseparability criteria for the linear polarization for $\alpha = \beta$ fixing three different lines for $Re[\alpha] = 0$, $Im[\alpha] = 0.3$ (continuous line), $Im[\alpha] = 0.7$ (dashed line), and $Im[\alpha] = 1.5$ (short-dashed line). (a) $\lambda_1/|\lambda_2| = 0$ and (b) $\lambda_1/|\lambda_2| = 0.1$. All plots indicate no entanglement in accordance with the separable, but not Gaussian, state obtained in this situation.

Thus the fact that one of the variances of one of the modes is squeezed bellow the noise limit is a good indicator that entanglement may be occurring in the circular polarization. Indeed if we plot the inseparability criterion for the circular polarization

$$I_{+,-}(\alpha,\beta) = \frac{1}{2}[\Delta^2(X_+ + X_-)(\alpha,\beta) \\ + \Delta^2(Y_+ - Y_-)(\alpha,\beta)], \qquad (55)$$

where the variances are obtained using the previous expressions for this polarization, we observe a clear signature of entanglement, as can be seen in Fig.5. We remark however that there are situations (parameter choices) where the criteria will not indicate entanglement, while the state is clearly entangled. This is not a surprising fact since the state is non-Gaussian and thus violation of criteria is not a necessary ingredient for the existence of entanglement. For comparison we plot in Fig.6 the one
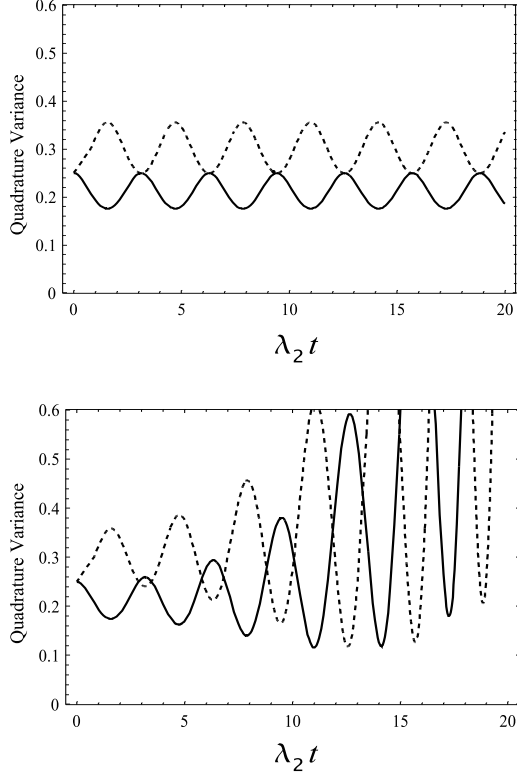
FIG. 4: Time evolution of quadrature variances for linearly polarized modes, with $\alpha = \beta$, $Re[\alpha] = 0$ and $Im[\alpha] = 0.3$. (a) $(\Delta X_y)^2$ (solid line) and $(\Delta Y_y)^2$ (dashed line) for $\lambda_1/|\lambda_2| = 0$. (b) $(\Delta X_y)^2$ (solid line) and $(\Delta Y_y)^2$ (dashed line) for $\lambda_1/|\lambda_2| = 0.1$. Mode x is left in a vacuum state.

mode reduced linear entropy, $S(\alpha, \beta) = 1 - Tr\{\rho_+^2\}$ time evolution, which in this case, since the joint system is pure, indicates entanglement between the modes. It is clear to notice that the criterion is correctly indicating entanglement for the same situations considered in Fig. 5.

It is interesting to remark that the points of Figs. 5 and 6 that indicate maximal entanglement are given by the state (51) for the circular polarization, which on its turn corresponds to a superposition of small amplitude coherent state (50) for the linear polarization. As we discussed this state has presented squeezing of one of its quadratures variance. Now we want to question how far is this state from the vacuum squeezed state, commonly attributed as being the case in many experimental situations? As a matter of fact, by following the discussion in [17], the fidelity, $F(|\xi\rangle, \rho_{cat}) = \sqrt{\langle \xi | \rho_{cat} | \xi \rangle}$, between this superposition state in the y polarization and the squeezed vacuum state,

$$|\xi\rangle = \frac{1}{\xi} \sum_{n=0}^{\infty} \left[\frac{\sqrt{(2n)!}}{n!} \left[-\frac{1}{2} e^{i\theta} \tanh(|\xi|) \right]^n \right] |2n\rangle, \quad (56)$$

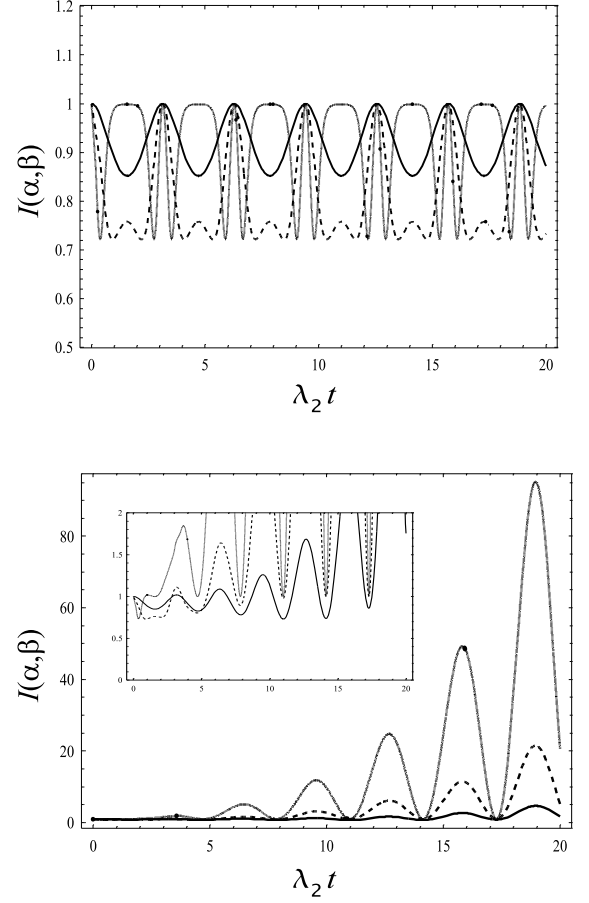can be very close to 1 as depicted in Fig. 7, and in some

FIG. 5: Plot of the inseparability criteria as a function of time for the circular polarization modes for $\alpha = \beta$ fixing three different lines for $Re[\alpha] = 0$, $Im[\alpha] = 0.3$ (solid line), $Im[\alpha] = 0.7$ (dashed line), and $Im[\alpha] = 1.5$ (short-dashed line). (a) $\lambda_1/|\lambda_2| = 0$. Curve points bellow 1 clearly signal entanglement of the circularly polarized modes (b) $\lambda_1/|\lambda_2| = 0.1$. The criterion is affected by the non-Hermiticity of the Hamiltonian, but at initial times it is still robust signaling entanglement, as depicted in the inset.

sense they are "similar" states. For small values of α (which represents the states generated in the mentioned experiment), the fidelity (with the squeezed vacuum state with small squeezing parameter) is very high. However, it is important to notice that the superposition state is *not* a Gaussian state, although the squeezed vacuum state *is* Gaussian. The fact of being or not Gaussian is important for the criterion of entanglement used to interpret the results in Josse *et al* [13]. Moreover the Entanglement of Formation [37] expression for symmetric Gaussian states, employed to quantify entanglement in Ref. [13], in this case is only a lower bound for the entanglement of the two modes.
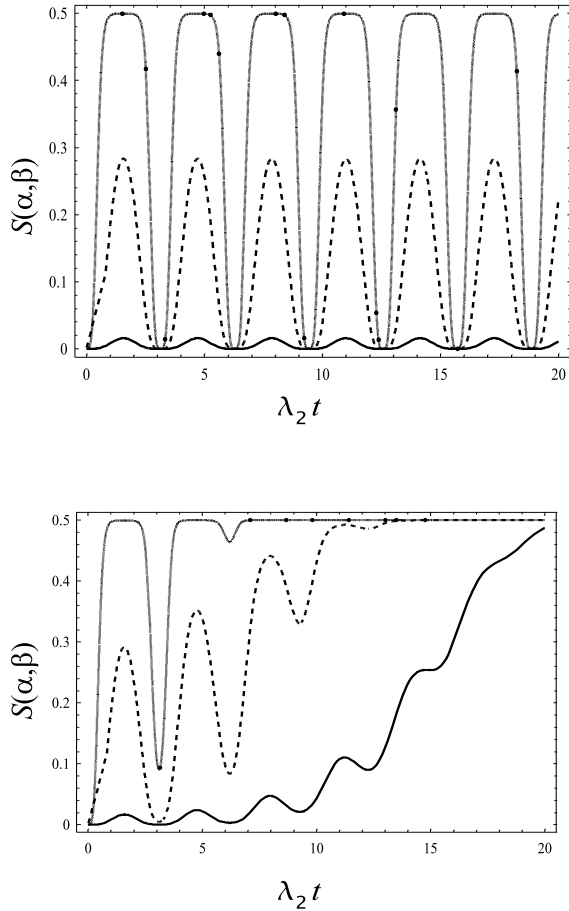
FIG. 7: (Color online) 3D plot of the fidelity between the generated state (Eq.50) and the squeezed vacuum state (Eq.56) as a function of α and ξ. The coupling condition adopted for the plot is $\lambda_1/|\lambda_2| \approx 0$.

FIG. 6: Plot of linear entropy time evolution for the $+$ circular polarization mode for $\alpha = \beta$ fixing three different lines for $Re[\alpha] = 0$, $Im[\alpha] = 0.3$ (continuous line), $Im[\alpha] = 0.7$ (dashed line), and $Im[\alpha] = 1.5$ (short-dashed line). (a) $\lambda_1/|\lambda_2| = 0$ Indicates a periodic entanglement and disentanglement of the two orthogonal modes. (b) $\lambda_1/|\lambda_2| = 0.1$ The periodicity of entanglement-disentanglement is affected by the non-Hermiticity of the Hamiltonian. The system tends to be highly entangled as the time goes on.

V. CONCLUSION

In this paper we have investigated a model [12] for the interaction of two quantum fields with an ensemble of X-like four level atoms, and we derived an effective Hamiltonian accounting for the field modes interaction. We have demonstrated that the Heisenberg equations for the field modes operators are non-linear leading to a non-Gaussian evolution. This non-linear evolution will lead to an entangled non-Gaussian state or to a non-Gaussian superposition of coherent states, when viewed from the circular or linear polarization reference frame, depending on the initial states for the two modes and atomic system. Even when the evolution is kept as close as possible from a Gaussian one, i.e., bilinear in the two field modes
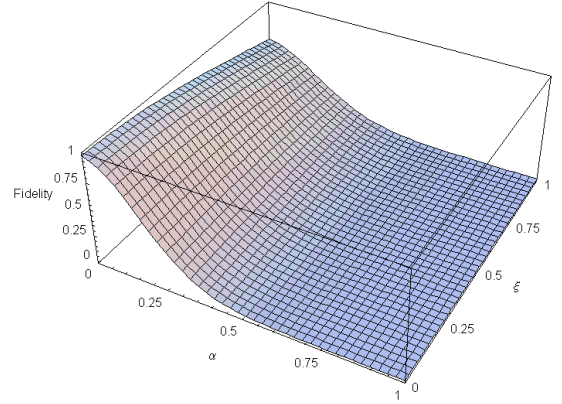
operators, a superposition state of two atomic degenerate fundamental collective states can lead to a non-Gaussian evolution. By appropriately setting one of the input linearly polarized modes in the vacuum state, we obtain in the output a coherent superposition of coherent states in the orthogonally polarized linear modes. Although this state is non-Gaussian it preserves similarities with the one-mode squeezed vacuum state. When viewed from the circular polarization frame, this superposition results in an entangled coherent state between the two modes in polarization $+$ and $-$. We have compared qualitatively these results to recent experimental results [13] with a similar system, which however have attributed a Gaussian nature to the Quantum fields. The presence of this superposition, in one polarization reference frame and an entangled state in the other can explain all the non-classical features observed in this experiment, with a non-Gaussian state however.

Linearization procedures for field operators are common in quantum optics whenever non-linear processes are present. While it can lead to a good approximation to non-classical features such as squeezing, it does not allow to correctly infer the non-Gaussian nature of the system. By linearizing one is at most mimicking the second order moments which are present in the most general nonlinear case. Thus it is commonly attributed a Gaussian evolution to the system as well, which as we have showed it is not particularly correct. Inference on quantum properties of single systems may well be described by the linearization procedure, but the description of entanglement of (at least) two modes suffers from inconsistencies which are only resolved when dealing with the correct nonlinearized case. One typical feature that cannot be correctly inferred is the separability of the two modes, since the criterion employed is only necessary and sufficient for Gaussian states. We have shown that invariably in the

present system the quantum state is non Gaussian. Remarkably recently much effort has been dedicated to the generation of such non-Gaussian states in propagating light fields by photon-subtraction [17, 18]. As we demonstrated, in principle such type of states may be generated in the experiments reported in [8, 12, 13, 14] and any other similar situation, whenever the atoms can be prepared in coherent superpositions if nonlinearities are to be neglected. If the nonlinearities are present, the entangled two mode field state will always be non-Gaussian independently of the atomic system state preparation.

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